

## SHEAVES OF G-STRUCTURES AND GENERIC G-MODELS

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**ABSTRACT.** In this article we give an equivariant version for the construction of generic models on presheaves of structures. We deal with first order structures endowed with a suitable action of some fixed group, say  $G$ ; we call them  $G$ -structures. We show that every exact presheaf of  $G$ -structures  $\mathcal{M}$  has a generic  $G$ -model  $\mathcal{M}^{\text{gen}}$ . Moreover,  $\mathcal{M}$  induces a presheaf of orbit structures  $\mathcal{M}/G$  and a generic orbit model  $(\mathcal{M}/G)^{\text{gen}} \cong \mathcal{M}^{\text{gen}}/G$  which coincides with the orbit structure of the generic  $G$ -model.

## FOREWORD

This article is inspired in both the work of Caicedo for topological sheaves of structures [4] and the geometric study of transformation groups [1] and is an attempt to further the connections between Model Theory and Geometry already opened in the work of many authors (see for instance Macintyre[14] for a survey of the connections and many open lines of work). We establish the first results towards a Model Theoretic analysis of geometric structures beyond sheaves. We combine the approach of Caicedo's for his Model Theory on Sheaves with the Geometric study of group actions.

Given a group  $G$ , a  $G$ -structure in a first order language  $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$  is a structure  $\mathcal{A} = (A, R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}})$  in the usual sense, such that  $G$  acts on the universe set  $A$  and the elements of  $G$  commute with the language symbols. Each  $G$ -structure  $\mathcal{A}$  induces an orbit structure  $\mathcal{A}/G$ . A morphism of  $G$ -structures is a morphism of structures which is  $G$ -equivariant. The category of  $G$ -structures is closed under colimits [11], and these behave well when passing to the orbit structures.

Given a topologic space  $X$ ; a presheaf  $\mathcal{M}$  of  $G$ -structures on  $X$  is one that maps each open nbhd  $U \subset X$  to some  $G$ -structure  $\mathcal{M}_U$ , and each inclusion  $U \subset V$  of

open nbhds to some morphism of  $G$ -structures  $\mathcal{M}_V \xrightarrow{p_{UV}} \mathcal{M}_U$ . Each presheaf of  $G$ -structures induces a presheaf of orbit structures. In this article we generalize the Generic Model Theorem 5.2 of [4] to the equivariant context. We achieve this by using only, and as far as we can, classical presheaf techniques, combined with semantics on sheaves and variants thereof. We study a version of the preservation of logical “truth”, under colimits and germs. Given a presheaf of  $G$ -structures  $\mathcal{M}$  on  $X$  we show that

- If  $\mathcal{M}$  has a generic  $G$ -model  $\mathcal{M}^{\text{gen}}$  then the presheaf of orbit structures  $\mathcal{M}/G$  has a generic orbit model  $(\mathcal{M}/G)^{\text{gen}} \cong \mathcal{M}^{\text{gen}}/G$  which coincides with the orbit structure of the generic  $G$ -model.
- If  $\mathcal{M}$  is exact, then it has a generic  $G$ -model.

Broadly speaking, these generalizations are obtained through a simplification in the presentation of the pointwise forcing relation in terms of presheaves.

There are several examples of presheaves of  $G$ -structures coming from a standard  $G$ -space and the usual functors of algebraic topology [1, 2, 3]; such as (a) Singular chains. (b) De Rham differential forms. (c) Usual homology and cohomology

with coefficients in  $\mathbb{Z}, \mathbb{R}$ . By the theorem of universal coefficients, homology and cohomology are essentially unique for modules without torsion. Some more interesting examples arise by allowing coefficients with torsion or geometric singularities; for instance, (d)  $q$ -homology coming from  $q$ -chains [6, 12]. (e) Inter-

section homology and cohomology [9, 18]. (f) Rational homotopy and homology

[20]. All of these are presheaves and, some of them, are actually sheaves. On each case, the induced orbit presheaf provides the corresponding  $G$ -invariant functor. In these cases, computing the (co)homology or homotopy modules involves trickier points. New examples of structures related to local and semilocal geometric information (i.e. foliations) have recently appeared in number theory [7]. Several new problems must be faced in order to consider (co)homology theories with non standard coefficients. We also do not know the relation between non standard (co)homology the standard corresponding ones.

Our paper has the following structure:

Section 1 provides the basic framework of sheaves, the way we will use them here. Section 2 starts the new constructions:  $G$ -structures. We provide the basic definitions of  $G$ -structures (roughly speaking, first order structures together with an action of the group  $G$  on them), morphisms, embeddings, etc., between them. We then study the preservation of validity and elementarity, and the connection between  $G$ -structures and their “orbit spaces” (quotients by the action of  $G$ ) in terms of morphisms, limits, colimits, etc. In section 3, we generalize usual sheaves of structures to  $G$ -structures and construct presheaves of  $G$ -structures and  $G$ -sheaves. We study the connection between the cocycle condition and the new definition  $G$ -exactness. The final sections (4, 5 and 6) complete the results of this paper: the definition of local forcing, construction of equivariant generic models and a generalization of the Generic Model Theorem to this setting.

## 1. SHEAVES

Recall the notion of a sheaf on a topological space  $(X, \mathcal{T})$  [3, 8].

1.1. A **presheaf** on  $X$  with values in a category  $\mathfrak{C}$  is a map  $\mathcal{F}$  which assigns an object  $\mathcal{F}(U) \in \mathfrak{C}$  for each open subset  $U \subset X$ , and a **restriction morphism**

$\mathcal{F}(V) \xrightarrow{\rho_{UV}} \mathcal{F}(U)$  for each inclusion of open subsets  $U \subset V$ ; in such a way that

- (1)  $\rho_{UU} = 1_{\mathcal{F}(U)}$  for each open subset  $U$  of  $X$ .
- (2)  $\rho_{UV}\rho_{VW} = \rho_{UW}$  for each open subsets  $U \subset V \subset W$ .

In other words,  $\mathcal{F}$  is a contravariant functor from the topology of  $X$  to  $\mathfrak{C}$ . A **morphism of (pre)sheaves**  $\mathcal{F} \xrightarrow{T} \mathcal{G}$  is a natural transformation from the functor  $\mathcal{F}$  to the functor  $\mathcal{G}$ . If the target categories are closed under inverse limits then  $T$  commutes with the limits whenever it makes sense.

A presheaf  $\mathcal{F}$  is a **sheaf** iff for any family of open subsets and their union

$$\{U_i : i \in \mathfrak{I}\} \subset \mathcal{T} \quad U = \bigcup_i U_i$$

the following conditions are satisfied:

- **Coherence**: If  $s, s' \in \mathcal{F}(U)$  are such that  $\rho_i(s) = \rho_i(s')$  coincide at  $\mathcal{F}(U_i)$  for each  $i$ ; then  $s = s'$ .
- **Exactness**: If some  $s_i \in \mathcal{F}(U_i)$  is given for each  $i \in \mathfrak{I}$  in such a way that  $\rho_{ij}(s_i) = \rho_{ij}(s_j)$  in  $\mathcal{F}(U_i \cap U_j)$  for each  $i, j \in \mathfrak{I}$  such that  $U_i \cap U_j \neq \emptyset$ ; then there is some  $s \in \mathcal{F}(U)$  such that  $\rho_i(s) = s_i$  in  $\mathcal{F}(U_i)$  for all  $i \in \mathfrak{I}$ .

## 2. G-STRUCTURES

For a given group  $G$  we introduce the notion of a  $G$ -structure.

2.1. A **language** or **signature** is a triple  $\mathcal{L} = (\mathcal{F}, \mathcal{R}, \mathcal{C})$  of sets; a set  $\mathcal{F}$  of **function symbols**, a set  $\mathcal{R}$  of **relation symbols** and a set  $\mathcal{C}$  of **constant symbols**. Each  $f \in \mathcal{F}$  (resp.  $R \in \mathcal{R}$ ) has an associated positive integer  $n_f$  (resp.  $n_R$ ) called its **arity**. Given a signature  $\mathcal{L}$ ; a **structure**  $\mathcal{M} = (M, \mathcal{F}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, \mathcal{C}^{\mathcal{M}})$  is a family of four sets satisfying

- (1)  $M \neq \emptyset$ ; it is the **universe** of the structure  $\mathcal{M}$ .
- (2) For each  $f \in \mathcal{F}$  there is a unique function  $M^{n_f} \xrightarrow{f^{\mathcal{M}}} M$  in  $\mathcal{F}^{\mathcal{M}}$ .
- (3) For each  $R \in \mathcal{R}$  there is a unique subset  $R^{\mathcal{M}} \subset M^{n_R}$  in  $\mathcal{R}^{\mathcal{M}}$ .
- (4) For each  $c \in \mathcal{C}$  there is a unique  $c^{\mathcal{M}} \in M$  in  $\mathcal{C}^{\mathcal{M}}$ .

A **morphism** of structures  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  is a function  $M \xrightarrow{\alpha} N$  between the respective universe sets; such that:

- (1)  $\alpha(f^{\mathcal{M}}(a)) = f^{\mathcal{N}}(\alpha(a))$  for each  $f \in \mathcal{F}$  and  $a \in M^{n_f}$ .
- (2)  $\alpha(R^{\mathcal{M}}) \subset R^{\mathcal{N}}$  for each  $R \in \mathcal{R}$ .
- (3)  $\alpha(c^{\mathcal{M}}) = c^{\mathcal{N}}$  for each  $c \in \mathcal{C}$ .

An **isomorphism** is a bijective morphism whose inverse map is also a morphism. We denote the category of structures with signature  $\mathcal{L}$  and its morphisms with the letter  $\mathfrak{M}$ .

A morphism  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  is said to be **saturated** iff  $\alpha^{-1} \left( R^{\mathcal{N}} \right) \subset R^{\mathcal{M}}$  for each relation symbol  $R \in \mathcal{R}$ . An **embedding** (resp. a **submersion**) is an injective (resp. surjective) saturated morphism. Given two structures  $\mathcal{M}, \mathcal{N}$  such that  $\mathcal{M} \subset \mathcal{N}$ ; we say that  $\mathcal{M}$  is a **substructure** of  $\mathcal{N}$  iff the inclusion map is an embedding, in that case we write  $\mathcal{M} \leq \mathcal{N}$ .

Given a saturated morphism  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$ ; there is a unique substructure  $\mathcal{J}(\alpha)$  of  $\mathcal{N}$  whose universe  $\text{Im}(\alpha)$  is the image set of  $\alpha$ . From the obvious equivalence relation on  $\mathcal{M}$  induced by  $\alpha$  we also obtain a quotient model  $\mathcal{M}/\sim$  whose universe is the quotient set  $\mathcal{M}/\sim$ ; the quotient projection  $\mathcal{M} \xrightarrow{q} \mathcal{M}/\sim$  is a submersion and the induced arrow  $\mathcal{M}/\sim \xrightarrow{\bar{\alpha}} \mathcal{J}(\alpha)$  is an isomorphism.

2.2. Recall the notions of formulas and validity [15]. A term in the language is a function symbol that can be obtained, starting from a finite set of free variables and language symbols, in a finite number of steps. An **atomic formula** is one of the form  $t(v) = s(v)$  or  $t(v) \in R$  where  $t(v), s(v)$  are terms and  $R$  is a relation symbol. A **formula** is a finite concatenation of atomic formulas and the usual logic symbols  $\wedge, \vee, \neg, \exists, \forall$ .

Given a formula  $\varphi(v)$  and  $a \in M^n$  we say that  $\mathcal{M}$  **models**  $\varphi(v)$  in  $a$ , and will write  $\mathcal{M} \models \varphi(a)$ ; whenever  $\varphi(a)$  is true. For each morphism  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  and each formula  $\varphi(v)$ ; we will say that  $\alpha$  **preserves the validity of**  $\varphi$  iff whenever  $a \in M^n$  and  $\varphi(a)$  makes sense, the following conditional holds

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(\alpha(a))$$

**Lemma 2.2.1. [Preservation of validity under morphisms]**

(1) *Morphisms commute with terms, i.e. given a morphism  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  and a term  $t(v)$  in  $n$  free variables, we have  $\alpha \left( t^{\mathcal{M}}(a) \right) = t^{\mathcal{N}}(\alpha(a))$  for each  $a \in M^n$ .*

(2) *Morphisms preserve the validity of formulas without  $\neg, \forall$ .*

(3) *Submersions preserve the validity of formulas without  $\neg$ .*

(4) *If*

(a)  *$\alpha$  is an isomorphism, or*

(b)  *$\alpha$  is an embedding and  $\varphi(v)$  is a formula without  $\forall, \exists$ ;*

*then  $\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(\alpha(a))$ .*

[Proof] By induction on formulas; see [15, p.11-14]. □

An **elementary embedding** is an embedding  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  such that, for each  $a \in M^n$  and each formula  $\varphi(v)$  the following equivalence holds

$$\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(\alpha(a))$$

For instance, by Lemma 2.2.1 all isomorphisms are elementary embeddings.

2.3. Fix some group  $G$ . A **weak G-structure** is a structure  $\mathcal{M}$  such that

- (1) The universe  $M$  is endowed with an action  $G \times M \xrightarrow{\Phi} M$ .
- (2) The action commutes with the language symbols. More precisely
  - (a) The set of constants is invariant:  $G\mathcal{C}^A = \mathcal{C}^A$ .
  - (b) Relations are invariant subsets:  $G\mathcal{R}^A = \mathcal{R}^A$  for each  $R \in \mathcal{R}$ .
  - (c) Functions preserve orbits: For each  $f \in \mathcal{F}$  with arity  $n$ ,  $g_1, \dots, g_n \in G$  and  $x_1, \dots, x_n \in A$ ; there is some  $g \in G$  such that

$$f^A(g_1 x_1, \dots, g_n x_n) = g f^A(x_1, \dots, x_n)$$

We say that  $\mathcal{M}$  is a **G-structure** if, additionally, functions are coordinatewise G-equivariant: This means that, for each  $f \in \mathcal{F}$  with arity  $n$ ,  $g_1, \dots, g_n \in G$  and  $x_1, \dots, x_n \in A$ ;

$$f^A(g_1 x_1, \dots, g_n x_n) = g_1 \dots g_n f^A(x_1, \dots, x_n)$$

2.4. Here there are some examples:

(1) A countable polyhedron can be seen as a structure  $\mathcal{M} = (\mathcal{S}, \subset)$  in the language of posets  $\mathcal{L} = (<)$ , where  $\mathcal{S}$  is a family of finite subsets in  $\aleph_0$  which is hereditary for subsets [17]: If  $u \subset v \in \mathcal{S}$  then  $u \in \mathcal{S}$ . A geometric realization  $K$  of  $\mathcal{M}$  can be obtained by setting each  $v \in \mathcal{S}$  to be the set of vertices of a face in  $K$ . An easy exercise is trying to find, for a given  $\mathcal{M}$ , the biggest subgroup  $G$  of the countable-order symmetric group  $S^{\aleph_0}$  such that  $\mathcal{M}$  is a  $G$ -structure. This  $G$  can be written as a direct sum of finite symmetric groups.

(2) Let  $G$  be a compact group and  $X$  a topologic  $G$ -space [1]. Then each  $g \in G$  provides, by left multiplication, a homeomorphism  $X \xrightarrow{g} X$  which, in turns, induces a chain isomorphism on singular chains

$$SC_*(X) \xrightarrow{g} SC_*(X).$$

It is easy to check that  $G$  acts linearly on the homology groups  $H_*(X)$ , so these are  $G$ -structures. When  $G$  is a Lie group and  $X$  is a smooth manifold; a similar construction can be done for the De Rham cohomology. This can be extended to more complicated (co)homology theories such as, for instance, Goresky and Macpherson's intersection homology [9], and its dual intersection cohomology [18].

2.5. We write  $GA = \{ga : g \in G, a \in A\}$  for  $A \subset M$ . Given  $a \in M$  the **orbit** of  $a$  is the subset  $\bar{a} = G\{a\}$ ; and the **isotropy** of  $a$  is the subgroup  $G_a = \{g \in G : ga = a\}$ . We write  $M/G$  for the set of orbits.

A **morphism** of  $G$ -structures is a  $G$ -equivariant morphism of structures. By a  **$G$ -substructure** of  $\mathcal{M}$  we mean a  $G$ -invariant substructure. The composition of  $G$ -equivariant morphisms (resp. embeddings, submersions, elementary embeddings) is again an arrow of this kind. The family of  $G$ -structures and  $G$ -equivariant morphisms (resp. embeddings, submersions, elementary embeddings) is a category, we will denote it by  $\mathfrak{M}_G$  (resp.  $\mathfrak{M}_G^{\leq}, \mathfrak{M}_G^{\geq}, \mathfrak{M}_G^{\prec}$ ).

Given a formula  $\varphi(v)$  with free variables  $v = (v_1, \dots, v_n)$  in  $\mathcal{L}$ ; we define its **lifting**  $\tilde{\varphi}(v)$  by induction. If  $\varphi(v)$  is  $t(v) = s(v)$  for some terms  $t, s$ ; then  $\tilde{\varphi}(v)$  is  $\exists g \in G[gt(v) = s(v)]$ . If  $\varphi(v)$  is  $t(v) \in R$  then  $\tilde{\varphi}(v)$  is  $\varphi(v)$ . Suppose that  $\alpha(v), \beta(v), \psi(v, w)$  are formulas in  $\mathcal{L}$  such that  $\tilde{\alpha}(v), \tilde{\beta}(v), \tilde{\psi}(v, w)$  have already been defined. If  $\varphi(v)$  is  $\alpha(v) \wedge \beta(v)$  (resp.  $\alpha(v) \vee \beta(v)$ ) then  $\tilde{\varphi}(v)$  is  $\tilde{\alpha}(v) \wedge \tilde{\beta}(v)$  (resp.  $\tilde{\alpha}(v) \vee \tilde{\beta}(v)$ ). If  $\varphi(v)$  is  $\exists w \psi(v, w)$  (resp.  $\forall w \psi(v, w)$ ) then  $\tilde{\varphi}(v)$  is  $\exists w \tilde{\psi}(v, w)$  (resp.  $\forall w \tilde{\psi}(v, w)$ ). Finally; if  $\varphi(v)$  is  $\neg \alpha(v)$  then  $\tilde{\varphi}(v)$  is  $\neg \tilde{\alpha}(v)$ .

The reader will notice that the previous is necessarily a notational convention rather than a formal definition: writing an expression such as  $\mathcal{M} \models \tilde{\varphi}$  a priori does not make sense, as the group  $G$  and its elements are not part of the original language  $\mathcal{L}$ . The meaning of  $\mathcal{M} \models \tilde{\varphi}(v)$  is of course the (informal!) notion “ $\mathcal{M} \models \varphi(v)$  holds of *some* element in the  $G$ -orbit of  $v$ , as witnessed by some  $g$ ”. For notational convenience, in this paper we keep the notation  $\mathcal{M} \models \tilde{\varphi}$ .

2.6. In what follows we will study some properties of  $G$ -structures, orbit structures and their limits. We will assume the following conventions: For an inverse system of  $G$ -structures  $\{\mathcal{M}_i : i \in \mathcal{D}\}$  and  $a \in \mathcal{M}_i$  for some  $i \in \mathcal{D}$  we write  $[a]$  for the germ of  $a$  in the colimit  $\mathcal{M} = \text{colim}_{i \in \mathcal{D}} \mathcal{M}_i$ . As we will show in this §; there is a well defined germ action of  $G$  on  $\mathcal{M}$ . We write  $\langle a \rangle$  for the orbit of  $[a]$  in the colimit structure.

**Proposition 2.6.1. [Properties of  $G$ -structures]**

- (1) Each a  $G$ -structure  $\mathcal{M}$  induces an orbit structure  $\mathcal{M}/G$ ; the orbit map  $\mathcal{M} \xrightarrow{\pi} \mathcal{M}/G$  is a submersion.
- (2) Each morphism of  $G$ -structures  $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$  induces a morphism in the orbit structures  $\mathcal{M}/G \xrightarrow{\bar{\alpha}} \mathcal{N}/G$ .
- (3) If  $\alpha$  is saturated (resp. an isomorphism, an embedding, a submersion) then so is  $\bar{\alpha}$ .
- (4) Given a formula  $\varphi(v)$  with free variables  $v = (v_1, \dots, v_n)$  in  $\mathcal{L}$ , and  $a \in \mathcal{M}^n$ ;  $\mathcal{M}/G \models \varphi(\bar{a}) \Leftrightarrow \mathcal{M} \models \tilde{\varphi}(a)$ .

[Proof] Properties (1), (2) and (3) are straightforward, we leave the details to the reader. We show property (4) by induction on formulas. If  $\varphi(v)$  is  $t(v) = s(v)$  then

$$\begin{aligned} \mathcal{M}/G \models \varphi(\bar{a}) &\Leftrightarrow [t^{\mathcal{M}/G}(\bar{a}) = s^{\mathcal{M}/G}(\bar{a})] \Leftrightarrow \overline{t^{\mathcal{M}}(a)} = \overline{s^{\mathcal{M}}(a)} \\ &\Leftrightarrow \exists g \in G [gt^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)] \Leftrightarrow \mathcal{M} \models \tilde{\varphi}(a) \end{aligned}$$

The proof when  $\varphi(v)$  is  $t(v) \in R$  is quite similar. Suppose that  $\varphi(v)$  is  $\alpha(v) \wedge \beta(v)$  and the statement holds for  $\alpha, \beta$ . Then

$$\begin{aligned} \mathcal{M}/G \models \varphi(\bar{a}) &\Leftrightarrow \mathcal{M}/G \models [\alpha(\bar{a}) \wedge \beta(\bar{a})] \\ &\Leftrightarrow \mathcal{M}/G \models \alpha(\bar{a}) \text{ and } \mathcal{M}/G \models \beta(\bar{a}) \\ &\Leftrightarrow \mathcal{M} \models \tilde{\alpha}(a) \text{ and } \mathcal{M} \models \tilde{\beta}(a) \Leftrightarrow \mathcal{M} \models [\tilde{\alpha}(a) \wedge \tilde{\beta}(a)] \\ &\Leftrightarrow \mathcal{M} \models \tilde{\varphi}(a) \end{aligned}$$

Again, the proof when  $\varphi(v)$  is  $\alpha(v) \vee \beta(v)$  is similar. When  $\varphi(v)$  is  $\exists w\psi(v, w)$ , then

$$\begin{aligned} \mathcal{M}/G \models \varphi(\bar{a}) &\Leftrightarrow \mathcal{M}/G \models \exists w\psi(\bar{a}, w) \Leftrightarrow \exists \bar{b}\psi(\bar{a}, \bar{b}) \\ &\Leftrightarrow \exists g_1, g_2 \in G \exists b\psi(g_1 a, g_2 b) \Leftrightarrow \exists g \in G \exists b\psi(ga, b) \end{aligned}$$

In the last step; for  $\Leftarrow$  pick  $g_1 = g$  and  $g_2 = 1$ . For  $\Rightarrow$ ; use the fact that functions and terms are coordinatewise  $G$ -equivariant (see §2.3-(2.c); so the  $g = g_1 g_2$ . Therefore,

$$\mathcal{M}/G \models \varphi(\bar{a}) \Leftrightarrow \exists b\tilde{\psi}(a, b) \Leftrightarrow \mathcal{M} \models \exists w\tilde{\psi}(a, w) \Leftrightarrow \mathcal{M} \models \tilde{\varphi}(a)$$

as desired. When  $\varphi(v)$  is  $\forall w\psi(v, w)$  one can proceed in the same way. Finally; when  $\varphi(v)$  is  $\neg\alpha(v)$  we have

$$\begin{aligned} \mathcal{M}/G \models \varphi(\bar{a}) &\Leftrightarrow \mathcal{M}/G \not\models \alpha(\bar{a}) \Leftrightarrow \forall g \in G, \mathcal{M} \not\models \alpha(ga) \\ &\Leftrightarrow \mathcal{M} \not\models \exists g \in G \alpha(ga) \Leftrightarrow \mathcal{M} \models \neg\tilde{\alpha}(a) \\ &\Leftrightarrow \mathcal{M} \models \tilde{\varphi}(a) \end{aligned}$$

This finishes the induction.  $\square$

**Proposition 2.6.2.**  $\mathfrak{M}_G, \mathfrak{M}_G^{\leq}, \mathfrak{M}_G^{\prec}$  and  $\mathfrak{M}_G^{\geq}$  are closed by colimits.

[Proof] We proceed by steps. Notice that

- $\mathfrak{M}_G$  is closed by limits: See [11, p.49-52]. Given an inverse system of structures

$(\mathcal{D}, \leq) \longrightarrow \mathcal{M}$ ; the colimit structure  $\mathcal{M} = \text{coLim}_{i \in \mathcal{D}} \mathcal{M}_i$  always exists. Its universe set  $M$  is the quotient of the disjoint union  $\bigsqcup_i M_i$  by the equivalence relation

$$\forall i \forall j \forall x \in M_i \forall y \in M_j (x \sim y \Leftrightarrow \exists k \leq i, j : (\rho_{k_i}(x) = \rho_{k_j}(y)))$$

Denote  $[x]$  the equivalence class of  $x \in M_i$ . For  $f \in \mathcal{F}$  define  $f^{\mathcal{M}}([x]) = [f^{\mathcal{M}_i}(x)]$ . Also, given  $R \in \mathcal{R}$  let  $R^{\mathcal{M}} = \text{coLim}_{i \in \mathcal{D}} R^{\mathcal{M}_i}$ . Finally, if  $c \in \mathcal{C}$  take  $c^{\mathcal{M}} = [c^{\mathcal{M}_i}]$  for any  $i \in \mathcal{D}$ . Notice that the quotient maps  $\mathcal{M}_i \xrightarrow{q_i} \mathcal{M}$  are morphisms.

•  $\mathfrak{M}_G$  is closed by limits: If the above is a directed system of  $G$ -structures with equivariant arrows; then there is a  $G$ -action on the universe  $M$  of  $\mathcal{M}$ , given by

$$g[x] = [gx]$$

and each quotient map  $q_i$  is  $G$ -equivariant. Next we show that functions are coordinatewise equivariant; we will do this only for the first coordinate. Let  $f \in \mathcal{F}$  be a function symbol with arity  $n$ . For each  $x_1, \dots, x_n \in M_i$  and  $g \in G$ ,

$$\begin{aligned} f^{\mathcal{M}}(g[x_1], \dots, [x_n]) &= f^{\mathcal{M}}([gx_1], \dots, [x_n]) = [f_i^{\mathcal{M}}(gx_1, \dots, x_n)] \\ &= [gf_i^{\mathcal{M}}(x_1, \dots, x_n)] = g[f_i^{\mathcal{M}}(x_1, \dots, x_n)] \\ &= gf^{\mathcal{M}}([x_1], \dots, [x_n]) \end{aligned}$$

In a similar way it is possible to check that the action commutes with relations and constants in  $\mathcal{M}$ .

•  $\mathfrak{M}_G^{\leq}, \mathfrak{M}_G^{\prec}$  and  $\mathfrak{M}_G^{\geq}$  are closed by limits: Since we put no condition on the restriction morphisms, the other subcategories are still closed under inverse limits.  $\square$

**Corollary 2.6.3.** *Each inverse system of  $G$ -structures  $\{\mathcal{M}_i : i \in \mathcal{D}\}$  induces an inverse system of orbit structures  $\{\mathcal{M}_i/G : i \in \mathcal{D}\}$ ; and*

$$\frac{\text{coLim}_{i \in \mathcal{D}} \mathcal{M}_i}{G} \cong \text{coLim}_{i \in \mathcal{D}} \left( \frac{\mathcal{M}_i}{G} \right)$$

[Proof] This is an easy consequence of 2.6.2 and 2.6.1-(3). Let  $\mathcal{M} = \text{coLim}_{i \in \mathcal{D}} \mathcal{M}_i$ ; it is enough to show that  $\mathcal{M}/G$  has the universal property of colimits. Since each quotient map  $\mathcal{M}_i \xrightarrow{q_i} \mathcal{M}$  is  $G$ -equivariant (i.e. a morphism of  $G$ -structures), it induces a well defined map on the orbit structures

$$\mathcal{M}_i/G \xrightarrow{\overline{q_i}} \mathcal{M}/G \quad \overline{x} \mapsto \langle x \rangle$$

where  $\langle x \rangle = \overline{[x]} \in \mathcal{M}/G$  is the orbit of the germ equivalence class  $[x] \in \mathcal{M}$ .  $\square$

*Remark 2.6.4.* By the above Lemma, we are allowed to use the following identification

$$\langle x \rangle = \overline{[x]} = [\overline{x}]$$



**Proposition 2.6.5.** *Let  $\mathcal{M} = \operatorname{colim}_{i \in \mathcal{D}} \mathcal{M}_i$  be a colimit of G-structures and  $a \in M_i^n$  for some  $i \in \mathcal{D}$ . If  $\varphi(v)$  has no  $\neg, \forall$ ; then  $\mathcal{M} \models \varphi([a])$  if and only if there is some  $j \leq i$  such that  $\mathcal{M}_j \models \varphi(\rho_{ji}(a))$ .*

[Proof] Let's show the implication  $(\Rightarrow)$ . If  $\mathcal{M} \models \varphi([a])$ ; then we can check, by induction, the following cases:

- (a)  $\varphi$  is  $t(v) = s(v)$ : If  $\mathcal{M} \models \varphi([a])$  then  $t^{\mathcal{M}}([a]) = s^{\mathcal{M}}([a])$  so  $\left[ t^{\mathcal{M}_i}(a) \right] = \left[ s^{\mathcal{M}_i}(a) \right]$ . There is some  $j \leq i$  such that  $\rho_{ji} \left( t^{\mathcal{M}_i}(a) \right) = \rho_{ji} \left( s^{\mathcal{M}_i}(a) \right)$ . Since  $\rho_{ji}$  is a morphism, by §2.2.1-(1) we get  $t^{\mathcal{M}_j}(\rho_{ji}(a)) = s^{\mathcal{M}_j}(\rho_{ji}(a))$ ; so  $\mathcal{M}_j \models \varphi(\rho_{ji}(a))$ .
- (b)  $\varphi(v)$  is  $t(v) \in R$ :  $\mathcal{M} \models \varphi([a])$  iff  $t^{\mathcal{M}}([a]) = \left[ t^{\mathcal{M}_i}(a) \right] \in R^{\mathcal{M}}$ . By step (4) in the proof of §2.6.2  $\exists j \leq i$  such that  $t^{\mathcal{M}_j}(\rho_{ji}(a)) = \rho_{ji} \left( t^{\mathcal{M}_i}(a) \right) \in R^{\mathcal{M}_j}$ , so  $\mathcal{M}_j \models \varphi(\rho_{ji}(a))$ .

So it holds for atomic formulas. Next we apply the inductive hypothesis for

- (c)  $\varphi(v)$  is  $\psi(v) \wedge \theta(v)$ : By induction, assume the statement for both  $\psi(v)$  and  $\theta(v)$ .  $\mathcal{M} \models \varphi([a])$  iff  $\mathcal{M} \models \psi([a])$  and  $\mathcal{M} \models \theta([a])$ . Then  $\exists k, k' \leq i$  such that  $\mathcal{M}_k \models \psi(\rho_{ki}(a))$  and  $\mathcal{M}_{k'} \models \theta(\rho_{k'i}(a))$ . Take any  $j \leq k, k'$  and  $\rho_{ji}(a) = \rho_{k'j}(\rho_{k'i}(a)) = \rho_{kj}(\rho_{ki}(a))$ . By §2.2.1-(4.b) we get  $\mathcal{M}_j \models \psi(\rho_{ji}(a))$  and  $\mathcal{M}_j \models \theta(\rho_{ji}(a))$ . Therefore,  $\mathcal{M}_j \models \varphi(\rho_{ji}(a))$ .
- (d)  $\varphi(v)$  is  $\psi(v) \vee \theta(v)$ : This is similar to the previous step.
- (e)  $\varphi(v)$  is  $\exists w \psi(v, w)$ :  $\mathcal{M} \models \varphi([a])$  iff there is some germ  $[b]$  such that  $\mathcal{M} \models \psi([a], [b])$ . By induction we can assume that there is some  $j \leq i$  such that  $b \in M_j^{n'}$  and  $\mathcal{M}_j \models \psi(\rho_{ji}(a), b)$ . Then  $\mathcal{M}_j \models \varphi(\rho_{ji}(a))$ .

This proves  $\Rightarrow$ ; the converse holds by §2.2.1-(2).  $\square$

**Corollary 2.6.6.** *In the hypothesis of Proposition 2.6.5;  $\mathcal{M}/G \models \varphi(\langle a \rangle)$  if and only if there is some  $j \leq i$  such that  $\mathcal{M}_j/G \models \varphi(\overline{\rho_{ji}(a)})$ .*

### 3. G-SHEAVES

3.1. A **presheaf of G-structures** on  $X$  is a presheaf  $\mathcal{T} \xrightarrow{\mathcal{M}} \mathfrak{M}_G$  in the sense of §1.1. Each open subset  $U$  of  $X$  is sent to some G-structure  $\mathcal{M}_U$  and each inclusion of open subsets  $U \subset V$  is mapped to the corresponding equivariant restriction morphism  $\mathcal{M}_V \xrightarrow{\rho_{UV}} \mathcal{M}_U$ . When the group  $G$  is trivial we obtain, in particular, a presheaf of structures as usual [4]. By §2.6.1-(2);

**Proposition 3.1.1.** *Each presheaf of  $G$ -structures induces a presheaf of orbit structures.*

Given a point  $x \in X$ , write  $\mathcal{V}(x)$  for the system of nbhds at  $x$ . Let  $\mathcal{M}_x = \text{colim}_{U \in \mathcal{V}(x)} \mathcal{M}_U$  be the colimit  $G$ -structure on the system of neighborhoods at  $x$ . Also, let  $[a]_x$  (resp.  $\langle a \rangle_x$ ) be the germ (resp. the orbit germ) of some  $a \in \mathcal{M}_U$  (resp. of  $\bar{a} \in \mathcal{M}_U/G$ ) for an open nbhd  $U \ni x$  and its corresponding  $G$ -structure  $\mathcal{M}_U$ . Proposition §2.6.5 can now be translated in terms of presheaves.

**Proposition 3.1.2.** *Let  $\mathcal{M}$  be a presheaf of  $G$ -structures on  $X$ ,  $x \in X$  a point,  $\varphi(v)$  a formula without  $\neg, \forall$  and  $a \in \mathcal{M}_V^n$ . Then  $\mathcal{M}_x/G \models \varphi(\langle a \rangle_x) \Leftrightarrow$  there is an open neighborhood  $U \ni x$  such that  $\mathcal{M}_U/G \models \varphi(\overline{\rho_{UV}(a)})$ .*

*Remark 3.1.3.* In the above situation,  $\mathcal{M}_y/G \models \varphi(\langle a \rangle_y)$  for all  $y \in U$ .

**3.2. A sheaf of  $G$ -structures** on  $X$  is a coherent and exact presheaf of  $G$ -structures §1.1. We write  $a|_U = \rho_{UV}(a)$  for the restriction of  $a \in \mathcal{M}_U$  to  $V \subset U$ . A  **$G$ -sheaf** is a sheaf of  $G$ -structures such that the quotient presheaf  $\mathcal{M}/G$  is still a sheaf. In general, a sheaf of  $G$ -structures is not a  $G$ -sheaf. For instance, let  $X = U \cup V$  be the union of two open subsets and take  $a, b \in \mathcal{M}_X$ . Suppose that  $\bar{a}|_U = \bar{b}|_U$  and  $\bar{a}|_V = \bar{b}|_V$ . Pick  $g_1, g_2 \in G$  such that

$$a|_U = (g_1 b)|_U \quad a|_V = (g_2 b)|_V$$

If  $g_1 = g_2 = g$  then, since  $\mathcal{M}$  is coherent,  $a = gb$ . Passing to the orbits,  $\bar{a} = \bar{b}$ . Nevertheless this is not always the case.

Given a presheaf  $\mathcal{M}$  of  $G$ -structures on  $X$  and an arbitrary union of open subsets  $U = \bigcup_{i \in \mathfrak{J}} U_i$  we say that

- $\mathcal{M}$  is  **$G$ -coherent** Iff given  $a, b \in \mathcal{M}_U$  and  $g_i \in G$  for each  $i \in \mathfrak{J}$  such that  $a = g_i b$  in  $U_i$  for all  $i$ ; then there is some  $g \in G$  such that  $a = gb$  in  $U$ .
- $\mathcal{M}$  is  **$G$ -exact** Iff given  $a_i \in \mathcal{M}_{U_i}$  and  $g_{ij} \in G$  for all  $i, j$  such that  $a_i = g_{ij} a_j$  in  $U_i \cap U_j$  for all  $i, j$ ; then there is  $a \in \mathcal{M}_U$  and there are  $h_i \in G$  for all  $i$  such that  $a = h_i a_i$  in  $U_i$  for all  $i$ .

**Lemma 3.2.1.** *A presheaf of  $G$ -structures is a  $G$ -sheaf  $\Leftrightarrow$  it is  $G$ -coherent and  $G$ -exact.*

[Proof] It is easy to check that  $\mathcal{M}$  is  $G$ -coherent  $\Leftrightarrow \mathcal{M}/G$  is coherent; and  $\mathcal{M}$  is  $G$ -exact  $\Leftrightarrow \mathcal{M}/G$  is exact.  $\square$

An intermediate condition between exactness and  $G$ -exactness is the following **cocycle condition**: Given a presheaf  $\mathcal{M}$  of  $G$ -structures on  $X$ , an arbitrary union of open subsets  $U = \bigcup_{i \in \mathfrak{J}} U_i$  as before,  $a_i \in \mathcal{M}_{U_i}$  and  $g_{ij} \in G$  for all  $i, j$  such that  $a_i = g_{ij} a_j$  in  $U_i \cap U_j$  for all  $i, j$ ; then  $g_{ik} = g_{ij} g_{jk}$  for all  $i, j, k$ .

**Lemma 3.2.2.** *An exact presheaf of  $G$ -structures satisfying the cocycle condition is  $G$ -exact.*

[Proof] It is enough to show that  $\mathcal{M}/G$  is exact. Let  $\mathcal{U} = \bigcup_{i \in \mathcal{J}} \mathcal{U}_i$  be a union of open subsets and fix  $\alpha_i \in \mathcal{M}_{\mathcal{U}_i}$  for each  $i$ ; such that their orbits satisfy  $\overline{\alpha}_i = \overline{\alpha}_j$  in  $\mathcal{U}_i \cap \mathcal{U}_j$  for all  $i, j \in \mathcal{J}$ . Then, by definition, we can pick for each  $i \leq j$  some  $g_{ij} \in G$  such that  $\alpha_i = g_{ij} \alpha_j$  on  $\mathcal{U}_i \cap \mathcal{U}_j$ . The main idea of the proof is the following: We will obtain from the  $\alpha_i$ 's a new set of  $b_i$ 's which will do the work.

By the axiom of choice we can assume that  $\mathcal{J}$  is well ordered. Define a function  $\phi$  on  $\mathcal{J}$  as follows: Let  $\phi(i)$  be the first  $k \leq i$  such that, for some positive integer  $r > 0$  there is a strict increasing finite chain of indexes  $k = k_1 < \dots < k_r = i$  in  $\mathcal{J}$  satisfying

$$\mathcal{U}_{k_s} \cap \mathcal{U}_{k_{s+1}} \neq \emptyset \quad s = 0, \dots, r-1$$

Let

$$b_i = g_{\phi(i), i} \alpha_i$$

Notice that  $\overline{b}_i = \overline{\alpha}_i$  for all  $i$ . For instance, if  $i_0$  is the first element of  $\mathcal{J}$  then  $\phi(i_0) = i_0$ . By the cocycle condition  $b_{i_0} = \alpha_{i_0}$ .

Next we show that the  $b_i$ 's coincide on the intersections. Take  $i < j$  in  $\mathcal{J}$  and assume that  $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ ; so  $\phi(i) = \phi(j) \leq i$  and, by the cocycle condition, also  $\alpha_i = g_{i,j} \alpha_j$ . Then

$$b_i = g_{\phi(i), i} \alpha_i = g_{\phi(j), i} \alpha_i = g_{\phi(j), i} (g_{i,j} \alpha_j) = (g_{\phi(j), i} g_{i,j}) \alpha_j = g_{\phi(j), j} \alpha_j = b_j$$

on  $\mathcal{U}_i \cap \mathcal{U}_j$ . Since  $\mathcal{M}$  is exact, there is some  $b \in \mathcal{M}_{\mathcal{U}}$  such that  $b = b_i$  on  $\mathcal{U}_i$  for all  $i \in \mathcal{J}$ . Passing to the orbit structures we get  $\overline{b} = \overline{b}_i = \overline{\alpha}_i$  on  $\mathcal{U}_i \cap \mathcal{U}_j$  for all  $i \in \mathcal{J}$ . We deduce that  $\mathcal{M}/G$  is exact.  $\square$

#### 4. LOCAL SEMANTICS

Next we recall the notions of point and open semantics on a presheaf of structures. These are natural continuous extensions of Propositions 2.6.5 and 3.1.2.

**4.1. Point semantics.** Fix some presheaf of  $G$ -structures  $\mathcal{M}$  on  $X$  and a point  $x \in X$ . Let  $\varphi(v)$  be a formula in free variables  $v = (v_1, \dots, v_n)$ . Given an open nbhd  $\mathcal{U} \ni x$  and some element  $a \in \mathcal{M}_{\mathcal{U}}$ ; we say that  $\mathcal{M}$  **forces**  $\varphi(a)$  at  $x$ , and we will write

$$\mathcal{M} \Vdash_x \varphi(a)$$

in the following cases:

- (1)  $\varphi(v)$  has no  $\neg, \forall$ : Iff there is some open nbhd  $\mathcal{U} \supset V \ni x$  such that  $\mathcal{M}_V \models \varphi(\rho_{\mathcal{U}V}(a))$ . Notice that, by Proposition 2.6.5 this is equivalent to require that  $\mathcal{M}_x \models \varphi([a]_x)$ .
- (2)  $\varphi(v)$  is  $\neg\psi(v)$ : Iff there is some open nbhd  $\mathcal{U} \supset V \ni x$  such that  $\mathcal{M} \not\models_y \psi(a)$  for all  $y \in V$ .
- (3)  $\varphi(v)$  is  $\psi(v) \rightarrow \nu(v)$ : Iff there is some open nbhd  $\mathcal{U} \supset V \ni x$  such that, for all  $y \in V$ ; if  $\mathcal{M} \models_y \psi(a)$  then  $\mathcal{M} \models_y \nu(a)$ .

(4)  $\varphi(v)$  is  $\forall w\psi(v, w)$ : Iff there is some open nbhd  $U \supset V \ni x$  such that, for each  $y \in V$  and each  $b \in \mathcal{M}_y$ , we have  $\mathcal{M} \Vdash_y \psi(a, b)$ .

**Proposition 4.1.1.** *On categoric sheaves of structures; the above definition of point semantics is equivalent to the one provided at [4] for topologic sheaves.*

[Proof] Let  $\mathcal{M}$  be a sheaf of structures,  $\xi = (E, p, X)$  the topologic sheaf induced by  $\mathcal{M}$ ; this is a sheaf of structures as defined at [4]. We must show that

$$(1) \quad \mathcal{M} \Vdash_x \varphi(a) \Leftrightarrow \xi \Vdash_x \varphi(\sigma)$$

for some local section  $\sigma$  defined at  $x$ . We will do this by induction on  $\varphi(v)$ . Let  $U \subset X$  be an open set. By [8, p.110],  $\mathcal{M}_U$  is the structure of local sections of  $\xi$  defined at  $U$ . Each local section  $\sigma$  defined at  $U$  is given in terms of some element  $a \in \mathcal{M}_U$  by the canonic representation map which sends each point  $y \in U$  to the germ of  $a$  in the colimit structure  $\mathcal{M}_y$ ; we will write this situation with the identity

$$\sigma(y) = [a]_y \quad \forall y \in U$$

If  $\varphi(v)$  is atomic then, by Proposition 2.6.5 and Definition 3.1-(1) at [4, p.15],

$$\mathcal{M} \Vdash_x \varphi(a) \Leftrightarrow \mathcal{M}_x \models \varphi([a]_x) \Leftrightarrow \mathcal{M}_x \models \varphi(\sigma(x)) \Leftrightarrow \xi \Vdash_x \varphi(\sigma)$$

as desired. Suppose that  $\varphi(v)$  is  $\alpha(v) \wedge \beta(v)$  and the statement holds for  $\alpha, \beta$ . If  $\xi \Vdash_x \varphi(\sigma)$  then, by the corresponding definition,  $\xi \Vdash_x \alpha(\sigma)$  and  $\xi \Vdash_x \beta(\sigma)$ . By induction,  $\mathcal{M} \Vdash_x \alpha(a)$  and  $\mathcal{M} \Vdash_x \beta(a)$ . Let  $V_\alpha, V_\beta \subset U$  be open nbhds of  $x$  such that  $\mathcal{M}_{V_\alpha} \models \alpha(\rho_{UV_\alpha}(a))$  and  $\mathcal{M}_{V_\beta} \models \beta(\rho_{UV_\beta}(a))$ . Take  $V = V_\alpha \cap V_\beta$ . Then, by Proposition 2.2.1-(2);  $\mathcal{M}_V \models \varphi(\rho_{UV}(a))$ ; so  $\mathcal{M} \Vdash_x \varphi(a)$ . This proves one implication, the converse is straightforward. If  $\varphi(v)$  is  $\alpha(v) \vee \beta(v)$  one can proceed in a similar way. The case when  $\varphi(v)$  is  $\exists w\psi(v, w)$  is straightforward. This proves the equivalence (1) above for any formula  $\varphi(v)$  without  $\neg, \forall$ . Conditions §4.1 (2),..., (5) are equivalent to their corresponding statements at Definition 3.1 (4),..., (7) in [4, p.16]. This finishes the proof.  $\square$

4.1.1. Given a point  $x \in X$ , an open nbhd  $U \ni x$ , a formula  $\varphi(v)$  in free variables  $v = (v_1, \dots, v_n)$  and some  $a \in \mathcal{M}_U^n$ ; the following properties hold:

- (a) **Local Semantics:**  $\mathcal{M} \Vdash_x \varphi(a)$  iff there is some open nbhd  $U \supset V \ni x$  such that  $\mathcal{M} \Vdash_y \varphi(a)$  for all  $y \in V$ .
- (b) **Classical Semantics:** For an isolated point  $x \in X$  we get  $\mathcal{M} \Vdash_x \varphi(a) \Leftrightarrow \mathcal{M}_x \models \varphi([a]_x)$ .
- (c) **Excluded Middle Principle:**  $\mathcal{M} \Vdash_x \forall u \forall v (u = v) \vee (u \neq v)$  iff (i)  $x$  has an open nbhd  $U_1$  such that, for all  $a, b \in \mathcal{M}_{U_1}$ ,  $a = b$ . Or, (ii)  $x$  has an open nbhd  $U_2$  such that, for all  $a, b \in \mathcal{M}_{U_2}$ ,  $a \neq b$ . When  $\mathcal{M}$  is a sheaf and  $X$  is Hausdorff, this means that the induced topologic sheaf is also Hausdorff in some nbhd of  $x$ .

**4.2. Open semantics.** Given a presheaf of G-structures  $\mathcal{M}$ , an open nbhd  $U \subset X$  and some  $a \in \mathcal{M}_U$ ; we say that  $\mathcal{M}$  **forces**  $\varphi(a)$  **in**  $U$ , and we write  $\mathcal{M} \Vdash_U \varphi(a)$ , iff  $\mathcal{M} \Vdash_x \varphi(a)$  for all  $x \in U$ . By §4.1.1-(1),

$$\mathcal{M} \Vdash_x \varphi(a) \Leftrightarrow \text{there is some nbhd } U \supset V \ni x \text{ such that } \mathcal{M} \Vdash_V \varphi(a)$$

**Lemma 4.2.1.** *Let  $\mathcal{M}$  be a G-sheaf on  $X$ ,  $\mathcal{M}/G$  the induced orbit sheaf and  $x \in X$  a point. Let  $\varphi(v)$  be a formula with free variables  $v = (v_1, \dots, v_n)$ ,  $\tilde{\varphi}(v)$  the lifting of  $\varphi(v)$ ,  $U \ni x$  an open nbhd and  $a \in \mathcal{M}_U^n$ . Then*

- (1)  $\mathcal{M}/G \Vdash_x \varphi(\bar{a}) \Leftrightarrow \mathcal{M} \Vdash_x \tilde{\varphi}(a)$ .
- (2)  $\mathcal{M}/G \Vdash_U \varphi(\bar{a}) \Leftrightarrow \mathcal{M} \Vdash_U \tilde{\varphi}(a)$ .

[Proof] (1) By §2.6.1-(4). (2) By definition and the first step of this proof.  $\square$

Also, the validity of  $\varphi(a)$  is related to the topology of  $X$  as follows:

- (a) Restrictions: If  $U \subset V$  then  $\mathcal{M} \Vdash_V \varphi(a) \Rightarrow \mathcal{M} \Vdash_U \varphi(a)$ .
- (b) Coverings:  $\mathcal{M} \Vdash_{U_i} \varphi(a|_{U_i}) \forall i \Rightarrow \mathcal{M} \Vdash_{\bigcup_i U_i} \varphi(a)$ .
- (c) Existencial quantifier:  $\mathcal{M} \Vdash_U \exists v \varphi(a, v)$  iff there is an open covering  $\bigcup_i U_i \supset U$  and some  $b_i \in \mathcal{M}_{U_i}$  for each  $i$ ; such that  $\mathcal{M} \Vdash_{U_i} \varphi(a, b_i)$  for each  $i$ .

**Proposition 4.2.2. [Maximum principle]** *Let  $\mathcal{M}$  be an exact presheaf of G-structures on  $X$ ,  $x \in X$  a point,  $U \ni x$  an open nbhd and  $a$  in  $\mathcal{M}_U$ . If  $\mathcal{M} \Vdash_U \exists v \varphi(a, v)$  then there is some open subset  $V \subset U$  and  $b \in \mathcal{M}_V$  such that  $U \subset \bar{V}$  and  $\mathcal{M} \Vdash_V \varphi(a, b)$ .*

[Proof] This is a traslation of Theorem 3.3 in [4, p.18]. Let  $\mathfrak{X} = \bigcup_{V \subset U} \mathcal{M}_V$  be the union of the universes of structures defined on open subsets of  $U$ ; this is, by our assumptions, a nonempty set. Consider in  $\mathfrak{X}$  the following partial order relation: For  $b \in \mathcal{M}_V$  and  $b' \in \mathcal{M}_W$  define

$$b \leq b' \Leftrightarrow V \subset W \text{ and } \rho_{VW}(b') = b$$

If  $\{a_i \in \mathcal{M}_{V_i} : i \in \mathbb{J}\}$  is a chain in  $\mathfrak{X}$  then, since  $\mathcal{M}$  is exact, for  $V = \bigcup_i V_i$  there is some  $a \in \mathcal{M}_V$  such that  $a_i \leq a$  for all  $i$ . Since  $V \subset U$  we get  $a \in \mathfrak{X}$ . By the Zorn Lemma; there is some maximal element  $a \in \mathcal{M}_W \subset \mathfrak{X}$ . By the maximality of  $a$ ,  $W$  is dense in  $U$ .  $\square$

## 5. EQUIVARIANT GENERIC MODELS

**5.1.** A **filter** in  $X$  is a family of open subsets  $\mathbb{F}$  which is closed by finite intersections and open supsets: For any  $U, V$  open in  $X$ ,

- (1)  $U, V \in \mathbb{F} \Rightarrow U \cap V \in \mathbb{F}$ .
- (2)  $V \in \mathbb{F}$  and  $U \supset V \Rightarrow U \in \mathbb{F}$

Notice that  $\mathbb{F}$  is trivial iff  $\emptyset \in \mathbb{F}$ . We say that  $\mathbb{F}$  is **maximal** iff it is not properly contained in any other filter. A straightforward application of the Zorn's Lemma

shows that there are maximal filters. A non trivial filter of open subsets  $\mathbb{F}$  in  $X$  is **generic with respect to  $\mathcal{M}$**  iff:

- (1) For each formula  $\varphi(v)$  in free variables  $v = (v_1, \dots, v_n)$ ,  $U \in \mathbb{F}$  and  $a \in M_U^n$ ; there is some  $U \supset V \in \mathbb{F}$  such that  $\mathcal{M} \Vdash_V \varphi(a)$  or  $\mathcal{M} \Vdash_V \neg \varphi(a)$ .
- (2) For each formula  $\varphi(v, w)$  in free variables  $v = (v_1, \dots, v_n)$  and  $w = (w_1, \dots, w_m)$ ; each  $U \in \mathbb{F}$  and  $a \in M_U^n$ ; if  $\mathcal{M} \Vdash_U \exists w \varphi(a, w)$  then there is some  $U \supset V \in \mathbb{F}$  and  $b \in M_V^m$  such that  $\mathcal{M} \Vdash_V \varphi(a, b)$ .

**Proposition 5.1.1. [Existence of generic filters]** *If  $\mathcal{M}$  is an exact presheaf of  $G$ -structures; then every maximal filter in  $X$  is generic with respect to  $\mathcal{M}$ .*

[Proof] For condition §5.1-(1) apply the same proof of Theorem 5.1 at [4, p.27]. For condition §5.1-(2) the main argument is the maximum principle which, in our context, only requires the hypothesis of exactness.  $\square$

**Proposition 5.1.2.** *Let  $\mathcal{M}$  be a presheaf of  $G$ -structures. Every generic filter with respect to  $\mathcal{M}$  is still generic with respect to  $\mathcal{M}/G$ .*

[Proof] We proceed by steps.

- (1) Pick some formula  $\varphi(v)$  and  $\bar{a}$  in  $M_U/G$  defined at some  $U \in \mathbb{F}$ . Since  $\mathbb{F}$  is generic with respect to  $\mathcal{M}$ ; there is some  $U \supset V \in \mathbb{F}$  such that  $\mathcal{M} \Vdash_V \tilde{\varphi}(\bar{a})$  or  $\mathcal{M} \Vdash_V \neg \tilde{\varphi}(\bar{a})$ , where  $\tilde{\varphi}$  is the lifting of  $\varphi(v)$ ; see §2.5. By Lemma 4.2.1,  $\mathcal{M}/G \Vdash_V \varphi(\bar{a})$  or  $\mathcal{M}/G \Vdash_V \neg \varphi(\bar{a})$ .
- (2) Pick some formula  $\varphi(v, w)$ ,  $U \in \mathbb{F}$  and  $\bar{a}$  in  $M_U/G$ . If  $\mathcal{M}/G \Vdash_U \exists w \varphi(\bar{a}, w)$ , by Lemma 4.2.1,  $\mathcal{M} \Vdash_U \exists w \tilde{\varphi}(\bar{a}, w)$ . Since  $\mathbb{F}$  is generic in  $\mathcal{M}$ ; there is some  $U \supset V \in \mathbb{F}$  and  $b \in M_V^m$  such that  $\mathcal{M} \Vdash_V \tilde{\varphi}(\bar{a}, b)$ . Again by Lemma 4.2.1,  $\mathcal{M}/G \Vdash_V \varphi(\bar{a}, \bar{b})$ .

Then  $\mathbb{F}$  is generic with respect to  $\mathcal{M}/G$ .  $\square$

5.2. A **generic model** for a presheaf of  $G$ -structures  $\mathcal{M}$  is the colimit structure

$$\mathcal{M}^{\text{gen}} = \text{coLim}_{U \in \mathbb{F}} \mathcal{M}_U$$

of the structures on a generic filter  $\mathbb{F}$  of  $X$ .

**Theorem 5.2.1. [Generic orbit models]** *If a presheaf  $\mathcal{M}$  has a generic  $G$ -model  $\mathcal{M}^{\text{gen}}$  then its orbit presheaf  $\mathcal{M}/G$  also has a generic orbit model, wich is the corresponding orbit model:*

$$\mathcal{M}^{\text{gen}}/G \cong (\mathcal{M}/G)^{\text{gen}}$$

The orbit map  $\mathcal{M}^{\text{gen}} \xrightarrow{\pi} (\mathcal{M}/G)^{\text{gen}}$  is, as usual, a submersion.

[Proof] Let  $\mathbb{F}$  be some filter in  $X$  which is generic with respect to  $\mathcal{M}$ . Then  $\mathcal{M}^{\text{gen}} = \text{coLim}_{U \in \mathbb{F}} \mathcal{M}_U$  is the colimit of the structures defined on open subsets in the filter. By Proposition 5.1.2  $\mathbb{F}$  is still generic with respect to  $\mathcal{M}/G$ . By Proposition

2.6.2,  $(\mathcal{M}/G)^{\text{gen}} = \text{coLim}_{\mathbb{U} \in \mathbb{F}} [\mathcal{M}_{\mathbb{U}}/G]$  is the generic model for  $\mathcal{M}/G$ . By Corollary 2.6.3 we get the isomorphism. The second statement is a consequence of Proposition 2.6.1-(1).  $\square$

**Corollary 5.2.2. [Existence of equivariant generic G-models]** *Every exact presheaf of G-structures  $\mathcal{M}$  has a generic G-model  $\mathcal{M}^{\text{gen}}$ ; its orbit presheaf  $\mathcal{M}/G$  also has a generic orbit model  $\mathcal{M}^{\text{gen}}/G \cong (\mathcal{M}/G)^{\text{gen}}$ .*

[Proof] By Propositions 5.1.1, 5.1.2 and Theorem 5.2.1.  $\square$

## 6. THE EQUIVARIANT GENERIC MODEL THEOREM

In the rest of this work we show how the models constructed at §5 are *generic*. Let us fix a presheaf of G-structures  $\mathcal{M}$  on  $X$ .

6.1. Let us show the behavior of the forcing relation under double negations. We start with two easy statements, the proofs are left to the reader who can go to [4] for more details.

**Lemma 6.1.1.** *Let  $\varphi(v)$  be a positive formula. Then  $\mathcal{M} \Vdash_{\mathbb{U}} \neg(\neg\varphi(a))$  iff there is some open nbhd  $V \subset \mathbb{U}$  such that  $V$  is dense in  $\mathbb{U}$  and  $\mathcal{M} \Vdash_V \varphi(a)$ .*

**Lemma 6.1.2.** *Let  $\mathbb{F}$  be a maximal filter of open nbhds in  $X$ , and  $\mathbb{U} \in \mathbb{F}$ . If  $V \subset \mathbb{U}$  is open and dense in  $\mathbb{U}$ , then  $V \in \mathbb{F}$ .*

The **Gödel translation**  $\varphi_{\mathbb{G}}$  of some formula  $\varphi$  is defined, by induction, as follows:

- $\varphi_{\mathbb{G}}$  is  $\neg(\neg\varphi)$  for an atomic formula  $\varphi$ .
- $(\varphi \wedge \psi)_{\mathbb{G}} = \varphi_{\mathbb{G}} \wedge \psi_{\mathbb{G}}$ .
- $(\varphi \vee \psi)_{\mathbb{G}} = \neg(\neg\varphi_{\mathbb{G}} \wedge \neg\psi_{\mathbb{G}})$ .
- $(\neg\varphi)_{\mathbb{G}} = \neg(\varphi_{\mathbb{G}})$ .
- $(\forall v\varphi)_{\mathbb{G}} = \forall v(\varphi_{\mathbb{G}})$ .
- $(\exists v\varphi)_{\mathbb{G}} = \neg\forall v(\neg\varphi_{\mathbb{G}})$ .

**Lemma 6.1.3.** *The Gödel translation commutes with the lifting,  $\widetilde{\varphi}_{\mathbb{G}} = \widetilde{\varphi_{\mathbb{G}}}$ .*

[Proof] As it is defined at §2.5; the lifting operator  $\varphi \mapsto \widetilde{\varphi}$  commutes with  $\wedge, \neg, \forall$ . The same is true for the Gödel translation  $\varphi \mapsto \varphi_{\mathbb{G}}$ ; notice that this operator is given, by induction, only in terms of  $\wedge, \neg, \forall$ .  $\square$

**Theorem 6.1.4. [Equivariant generic model theorem]** *Let  $\mathcal{M}$  be a sheaf of G-structures on  $X$  and  $\mathcal{M}^{\text{gen}}$  the generic model induced by some generic filter  $\mathbb{F}$  on  $X$ . For each formula  $\varphi(v)$ ,  $\mathbb{U} \in \mathbb{F}$  and  $a \in \mathcal{M}_{\mathbb{U}}$ ; the following statements are equivalent:*

- (1)  $\mathcal{M}^{\text{gen}} \models \widetilde{\varphi}([a])$ .
- (2)  $\mathcal{M}^{\text{gen}}/G \models \varphi(\langle a \rangle)$ .
- (3)  $\mathcal{M} \Vdash_V \widetilde{\varphi}_{\mathbb{G}}(a)$  for some  $\mathbb{U} \supset V \in \mathbb{F}$ .
- (4)  $\mathcal{M}/G \Vdash_V \varphi_{\mathbb{G}}(\overline{a})$  for some  $\mathbb{U} \supset V \in \mathbb{F}$ .



$$(5) \{x \in \mathcal{U} : \mathcal{M} \Vdash_x \varphi_{\mathbb{G}}(a)\} \in \mathbb{F}.$$

[Proof] (1)  $\Leftrightarrow$  (2) and (3)  $\Leftrightarrow$  (4) follow from Proposition 2.6.1 and Theorem 5.2.1. We will prove the other implications one step at a time.

(1)  $\Leftrightarrow$  (3): Let us proceed by induction.

- $\varphi(v)$  is atomic: By Proposition §2.6.5,  $\mathcal{M}^{\text{gen}} \models \tilde{\varphi}([a])$  iff there is some open nbhd  $\mathcal{U} \supset V \in \mathbb{F}$  such that  $\mathcal{M}_{\mathcal{U}} \models \tilde{\varphi}(\rho_{\mathcal{U}V}(a))$ . The definition of local forcing §4.2, implies that  $\mathcal{M} \Vdash_V \tilde{\varphi}(a)$ . By the definition of the lifting §2.5 and Lemma 6.1.3, we get  $\tilde{\varphi}_{\mathbb{G}} = \neg(\neg\tilde{\varphi})$ . Finally, by Lemma 6.1.1,  $\mathcal{M} \Vdash_V \tilde{\varphi}_{\mathbb{G}}(a)$ . Conversely; if  $\mathcal{M} \Vdash_V \tilde{\varphi}_{\mathbb{G}}(a)$  for some  $V \in \mathbb{F}$  then there is some open nbhd  $W \subset V$  such that  $W$  is dense in  $V$  and  $\mathcal{M} \Vdash_W \tilde{\varphi}(a)$ . By Lemma 6.1.2,  $W \in \mathbb{F}$  and, by Proposition §2.6.5,  $\mathcal{M}^{\text{gen}} \models \varphi([a])$ .
- $\varphi(v)$  is  $\alpha(v) \wedge \beta(v)$ :  $\mathcal{M}^{\text{gen}} \models \tilde{\varphi}([a])$  iff  $\mathcal{M}^{\text{gen}} \models \tilde{\alpha}([a])$  and  $\mathcal{M}^{\text{gen}} \models \tilde{\beta}([a])$ . By induction; this happens iff there are open nbhds  $\mathcal{U} \supset V_1, V_2$  in  $\mathbb{F}$  satisfying  $\mathcal{M} \Vdash_{V_1} \tilde{\alpha}_{\mathbb{G}}(a)$  and  $\mathcal{M} \Vdash_{V_2} \tilde{\beta}_{\mathbb{G}}(a)$ . By Lemma 6.1.3;

$$\tilde{\alpha}_{\mathbb{G}} \wedge \tilde{\beta}_{\mathbb{G}} = \left( \widetilde{\alpha \wedge \beta} \right)_{\mathbb{G}} = \tilde{\varphi}_{\mathbb{G}}$$

Then  $V = (V_1 \cap V_2) \in \mathbb{F}$  and  $\mathcal{M} \Vdash_V \tilde{\varphi}_{\mathbb{G}}(a)$ , as desired. The converse is straightforward.

- $\varphi(v)$  is  $\neg\alpha(v)$ :  $\mathcal{M}^{\text{gen}} \models \tilde{\varphi}([a])$  iff  $\mathcal{M}^{\text{gen}} \not\models \tilde{\alpha}([a])$ . By induction;  $\mathcal{M}_V \not\models \tilde{\alpha}_{\mathbb{G}}(a)$  for all  $\mathcal{U} \supset V \in \mathbb{F}$ , let us pick just one  $V$  from these. Since  $\tilde{\varphi}_{\mathbb{G}} = \widetilde{(\neg\alpha)}_{\mathbb{G}} = \neg(\tilde{\alpha}_{\mathbb{G}})$  by Lemma 6.1.3; we get that  $\mathcal{M}_V \models \tilde{\varphi}_{\mathbb{G}}$  as desired.
- $\varphi(v)$  is  $\forall v\alpha(v)$ : Proceed as in the previous step.

The other inductive steps are easy to check; we leave them to the reader. Finally, the equivalence (3)  $\Leftrightarrow$  (5) can be seen as before, with an inductive proof. It is, essentially, a consequence of Lemmas 6.1.1 and 6.1.2.  $\square$

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